Constrained Quantization of Charged Strings in Background B Field and g-Factors

Akira Kokado[*], Gaku Konisi[†] and Takesi Saito[‡]

Kobe International University, Kobe 655-0004, Japan[*]

Department of Physics, Kwansei Gakuin University, Nishinomiya 662-8501, Japan[†,‡]

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Abstract

The Dirac quantization is performed for the constrained system of the open string with different charges located at both ends in the constant background B field. Noncommutativity reveals to commutators $[X, X]$, $[P, P]$ and also $[X, P]$ at both ends of the string. We consider a dependence on the change of the "cyclotron frequency" of the charged string. The g-factor of the charged string is also calculated in the framework of our formulation.

[*]E-mail: kokado@kobe-kiu.ac.jp
[†]E-mail: konisi@kgupyr.kwansei.ac.jp
[‡]E-mail: tsaito@yukawa.kyoto-u.ac.jp

I. Introduction

The idea of noncommutative structure at small length scales is not new. In 1947 Snyder first considered "quantized space-time." The idea was that noncommutative space-times could introduce an effective cut-off in field theory similar to a lattice. Recently there has been a revival of this idea and published many papers[1]. Especially interesting is the model of open strings propagating in a constant two-form (B field) background. Previous studies show that this model is related to noncommutativity of D-branes[2,3], and in the zero slope limit to noncommutative Yang-Mills theory[4]. The equivalence of the ordinary gauge fields and the noncommutative gauge fields has then been proposed[5]. Recently the equivalence has been discussed in string field theories based on the noncommutative algebra[6]. An intriguing mixing of the UV and IR has also been found in the perturbative dynamics of noncommutative field theories[7].

Canonical quantization of the open string in the presence of D-branes with a constant B field has so far been discussed in the literature[8], treating the mixed boundary conditions as primary constraints and employing the Dirac quantization method. This is, however, restricted to quantization of the neutral string with opposite charges at both ends. In the present paper we would like to generalize this quantization to the case of the open string with different charges at both ends (charged string) in the B field. The spectrum of charged open strings in the B field was first considered in Ref.[9], and recently in Ref.[10]. In these works, however, the Dirac quantization had not yet been considered. This generalization is important, though it is rather exhausting.

In Secs. II-V the Dirac brackets are calculated. In Sec. VI we find noncommutativity for X and X, and also for P and P at both ends of the charged string, where P is the canonical momentum of X. We also find that there is a noncommutative term in the commutator $[X, P]$ except for the delta function. In the remaining sections we consider several topics: VII. Mode expansions, VIII. Virasoro algebra, IX. Dependence on the "cyclotron frequency", X. The g-factors of charged strings, XI. Conclusions.

The g-factors of charged strings have already been caluculated in the old ordinary formulation[11]. Here, we would like to calculate them again more completely in the framework of our formulation, which leads us to the noncommutative strings.

II. Lagrangian

Let the electromagnetic field couple to charges q_0 and q_π at the ends of the open string. The interaction Lagrangian is given by

$$L_I = q_0 \dot{X}^\mu A_\mu(\sigma = 0) + q_\pi \dot{X}^\mu A_\mu(\sigma = \pi). \tag{2.1}$$

We choose a gauge, $A_\mu = -(1/2)B_{\mu\nu}X^\nu$, where $B_{\mu\nu}$ is a constant background field. By introducing a function $\rho(\sigma)$ such that

$$\rho(0) = q_0, \quad \rho(\pi) = -q_\pi, \tag{2.2}$$

Eq.(2.1) can be written as

$$
\begin{aligned}
L_I &= \frac{1}{2}\rho(\sigma)\dot{X}BX(\sigma)\mid_{\sigma=0}^{\sigma=\pi} = \frac{1}{2}\int_0^\pi d\sigma\partial_\sigma\{\rho(\sigma)\dot{X}BX(\sigma)\} \\
&= \frac{1}{2}\int_0^\pi d\sigma(\rho'\dot{X}BX + \rho\dot{X}'BX + \rho\dot{X}BX') \\
&= \frac{1}{2}\int_0^\pi d\sigma\{\rho'\dot{X}BX + 2\rho\dot{X}BX' + \partial_\tau(\rho X'BX)\}.
\end{aligned} \tag{2.3}
$$

Dropping the last total derivative term, we have the total Lagrangian

$$
L = \frac{1}{4}\int_0^\pi d\sigma\{\dot{X}^2 - X'^2 + 2\rho\dot{X}BX' + \rho'\dot{X}BX\}. \tag{2.4}
$$

where the coefficient is chosen as 1/4 rather than the conventional 1/2 so that the final results coincide with those of the conventional (*i.e.* non-Dirac) quantization. (See Chap. IV and Appendix B.) Our system does not depend on the functional form of $\rho(\sigma)$ for $0 < \sigma < \pi$. This can be seen from the fact that the action based on (2.4) is invariant under a variation with respect to $\rho(\sigma)$. The equation of motion and the boundary conditions follow from (2.4)

$$
\ddot{X} - X'' = 0, \tag{2.5}
$$

$$
(X' + \rho B\dot{X})\mid_{0,\pi} = 0. \tag{2.6}
$$

The canonical conjugate momentum is given by

$$
P_c = \frac{1}{2}(\dot{X} + \rho BX' + \frac{1}{2}\rho' BX). \tag{2.7}
$$

Let us define an antisymmetric tensor $\beta^\mu{}_\nu(\sigma)$ by

$$
\rho(\sigma)B^\mu{}_\nu = \{\tanh\beta(\sigma)\}^\mu_\nu, \tag{2.8}
$$

and

$$
\beta_0 = \beta(0), \quad \beta_\pi = \beta(\pi), \tag{2.9}
$$

$$
\rho(0)B = \tanh\beta_0, \qquad \rho(\pi)B = \tanh\beta_\pi. \tag{2.10}
$$

We shall choose $\beta(\sigma)$ to be linear in σ as

$$
\beta(\sigma) = \beta_0 - \gamma\sigma, \quad \gamma \equiv \frac{1}{\pi}(\beta_0 - \beta_\pi). \tag{2.11}
$$

The $\beta(\sigma)$ can be regarded as a "rotational angle" between $(\dot{X}, X')$ and (P, Q), as is seen from Eqs.(7.5)-(7.8). Hence, the Virasoro operator (8.1) does not depend on $\beta(\sigma)$, as a result. β_0, β_π, $\beta(\sigma)$ and γ are all functions of B and commutable with each other. In the following we extend the region of σ to $(-\infty, \infty)$, but the physical region is still $(0, \pi)$.

III. Constraints

Define $\phi_l(\sigma)$, $\psi_l(\sigma)$ $(l = 0, \pi)$ by

$$\phi_l(\sigma) \equiv \cosh\beta_l \cdot X'(\sigma) + \sinh\beta_l \cdot \dot{X}(\sigma), \qquad (l = 0, \pi) \tag{3.1}$$
$$\psi_l(\sigma) \equiv \sinh\beta_l \cdot X'(\sigma) + \cosh\beta_l \cdot \dot{X}(\sigma). \qquad (l = 0, \pi) \tag{3.2}$$

The boundary conditions (2.6) are expressed as

$$\phi_l(\sigma_l) \equiv \cosh\beta_l \cdot X'(\sigma_l) + \sinh\beta_l \cdot \dot{X}(\sigma_l) = 0. \qquad (\sigma_0 = 0, \sigma_\pi = \pi) \tag{3.3}$$

These are primary constraints. From the consistency postulate

$$(\phi_l(\sigma_l), H)_{P.B} = 0, \tag{3.4}$$

we have the secondary constraints

$$\begin{aligned} \cosh\beta_l \cdot \dot{X}'(\sigma_l) + \sinh\beta_l \cdot \ddot{X}(\sigma_l) &= \cosh\beta_l \cdot \dot{X}'(\sigma_l) + \sinh\beta_l \cdot X''(\sigma_l) \\ &= \psi_l'(\sigma_l) = 0. \end{aligned} \tag{3.5}$$

In the same way we get

$${\phi_l}^{2n}(\sigma_l) = {\psi_l}^{2n+1}(\sigma_l) = 0. \quad (n = 0, 1, 2, \cdots) \tag{3.6}$$

Eqs.(3.6) show that $\phi_l(\sigma)$ $(\psi_l(\sigma))$ is the odd (even) function of $\sigma - \sigma_l$, that is,

$$\phi_l(2\sigma_l - \sigma) = -\phi_l(\sigma), \quad \psi_l(2\sigma_l - \sigma) = \psi_l(\sigma). \tag{3.7}$$

Let us write Eqs.(3.7) as

$$(1 - \Delta_l)\psi_l = (1 + \Delta_l)\phi_l = 0, \tag{3.8}$$

where

$$(\Delta_l f)(\sigma) = f(2\sigma_l - \sigma). \tag{3.9}$$

Eqs.(3.8) are equivalent to one equation

$$\begin{aligned} (1 - \Delta_l)\psi_l + (1 + \Delta_l)\phi_l &= \psi_l + \phi_l - \Delta_l(\psi_l - \phi_l) \\ &= e^{\beta_l}(\dot{X} + X') - e^{-\beta_l}(\dot{X} - X') = 0. \end{aligned} \tag{3.10}$$

Useful formulas are

$${\Delta_l}^2 = 1 \tag{3.11}$$

and

$$\boldsymbol{I} f(\sigma) = \Delta_0 f(2\pi - \sigma) = f(2\pi + \sigma), \tag{3.12}$$

where

$$\boldsymbol{I} \equiv \Delta_0\Delta_\pi, \qquad \boldsymbol{I}^{-1} = \Delta_\pi\Delta_0, \tag{3.13}$$

I being the 2π-displacement operator.

IV. Poisson brackets

The Dirac quantization will result in the commutator $[X(\sigma), P_c(\sigma')]$ to be (1/2) $\delta(\sigma-\sigma')$ for $0<\sigma,\sigma'<\pi$. This means that the correct momentum (the translation generator) is not P_c but

$$P \equiv 2P_c = \dot{X} + \rho B X' + \frac{1}{2}\rho' B X. \tag{4.1}$$

for which the Poisson bracket is given as

$$(X(\sigma), P(\sigma'))_{PB} = 2\delta(\sigma-\sigma'). \tag{4.2}$$

This is the reason why we have chosen the factor 1/4 in the Lagrangian (2.4).

Let us define $P_\pm$ by

$$P_\pm = \frac{1}{2}(\dot{X} \pm X') = \frac{1}{2}(\hat{P} - \rho B X' \pm X') \tag{4.3}$$

with

$$\hat{P} \equiv P - \frac{1}{2}\rho' B X. \tag{4.4}$$

Then Eq.(3.10) can be written as

$$\chi \equiv P_+ - e^{-2\beta_l}\Delta_l P_- = 0. \tag{4.5}$$

Poisson brackets for X, X' and $\hat{P}$ are

$$(X(\sigma), \hat{P}(\sigma'))_{PB} = 2\delta(\sigma-\sigma'), \tag{4.6}$$

$$(X'(\sigma), \hat{P}(\sigma'))_{PB} = 2\delta'(\sigma-\sigma'), \tag{4.7}$$

$$(\hat{P}(\sigma), \hat{P}(\sigma'))_{PB} = -2\rho'(\sigma) B \delta(\sigma-\sigma'), \tag{4.8}$$

where δ is the ordinary (non-periodic) delta function. We express (4.6), (4.7) and (4.8) symbolically as

$$(X, \hat{P})_{PB} = 2, \tag{4.9}$$

$$(X', \hat{P})_{PB} = 2\partial, \tag{4.10}$$

$$(\hat{P}, \hat{P})_{PB} = -2\rho' B = -2\beta' \cdot \mathrm{sech}^2\beta. \tag{4.11}$$

From these Eqs. we have

$$(P_\pm, P_\pm)_{PB} = \pm\partial, \quad (P_\pm, P_\mp)_{PB} = 0. \tag{4.12}$$

Noting

$$\partial\Delta_l = -\Delta_l\partial, \tag{4.13}$$

we obtain

$$
\begin{aligned}
G_{ll'} &\equiv (\chi_l, \chi_{l'})_{PB} = \partial - e^{-2\beta_l+2\beta_{l'}}\Delta_l \partial \Delta_{l'} = (1 + e^{-2\beta_l+2\beta_{l'}}\Delta_l\Delta_{l'})\partial \\
&\equiv H_{ll'}\partial,
\end{aligned}
\tag{4.14}
$$

where

$$
H \equiv \begin{pmatrix} 2 & 1+e^{-2\pi\gamma}\boldsymbol{I} \\ 1+e^{2\pi\gamma}\boldsymbol{I}^{-1} & 2 \end{pmatrix}. \quad \beta_0 - \beta_\pi = \pi\gamma \tag{4.15}
$$

In order to define the Dirac bracket, we need the inverse of G in its non-singular subspace. In the subspace $\boldsymbol{I} = e^{2\pi\gamma}$ the rank of H is reduced, that is, H becomes

$$
H = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1, & 1 \end{pmatrix} = 4uu^T, \quad u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.16}
$$

Hence, defining the projection operator to the subspace $I = e^{2\pi\gamma}$ by Λ_γ, and $\bar{\Lambda}_\gamma \equiv 1 - \Lambda_\gamma$, we have

$$
G^{-1} = \partial^{-1}(\bar{\Lambda}_\gamma H^{-1} + \frac{1}{4}\Lambda_\gamma uu^T) = (\bar{\Lambda}_\gamma H^{-1} + \frac{1}{4}\Lambda_\gamma uu^T)\partial^{-1}. \tag{4.17}
$$

The first term is written as

$$
\bar{\Lambda}_\gamma H^{-1} = \bar{\Lambda}_\gamma D_\gamma^{-1}\begin{pmatrix} 2 & -1-e^{-2\pi\gamma}\boldsymbol{I} \\ -1-e^{2\pi\gamma}\boldsymbol{I}^{-1} & 2 \end{pmatrix}, \tag{4.18}
$$

where

$$
D_\gamma = 4 - (1+e^{-2\pi\gamma}\boldsymbol{I})(1+e^{2\pi\gamma}\boldsymbol{I}^{-1}) = (1-e^{-2\pi\gamma}\boldsymbol{I})(1-e^{2\pi\gamma}\boldsymbol{I}^{-1}). \tag{4.19}
$$

There is a constant term in ∂^{-1}

$$
\langle\sigma|\partial^{-1}|\sigma'\rangle = \frac{1}{2}\epsilon(\sigma-\sigma') + c, \tag{4.20}
$$

where $\epsilon(\sigma) = \mathrm{sgn}(\sigma)$.

The Dirac bracket is then given by

$$
(A,B)_{DB} = (A,B)_{PB} - \sum_{l=0,\pi}\sum_{l'=0,\pi}(A,\chi_l)_{PB}(G^{-1})_{ll'}(\chi_{l'},B)_{PB}. \tag{4.21}
$$

Δ_l and $\boldsymbol{I}$ satisfy the following relations:

$$
\Delta_0\boldsymbol{I} = \Delta_\pi, \qquad \boldsymbol{I}\Delta_\pi = \Delta_0, \qquad \Delta_\pi\boldsymbol{I}^{-1} = \Delta_0, \qquad \boldsymbol{I}^{-1}\Delta_0 = \Delta_\pi, \tag{4.22}
$$

$$
\Delta_l\boldsymbol{I} = \boldsymbol{I}^{-1}\Delta_l. \tag{4.23}
$$

From (4.23) one can see for the projction Λ_γ

$$
\Delta_l\Lambda_\gamma = \Lambda_{-\gamma}\Delta_l. \tag{4.24}
$$

V. Dirac brackets

Here we use $\beta(\sigma)$ defined by (2.11). The operation Δ_l to $\beta(\sigma)$ gives

$$\Delta_l \beta(\sigma) = \beta(2\sigma_l - \sigma) = 2\beta_l - \beta(\sigma). \tag{5.1}$$

Since from (4.6) we have

$$(X, P_\pm)_{PB} = 1, \tag{5.2}$$

$$(\hat{P}, P_\pm)_{PB} = -\rho' B + \partial \rho B \pm \partial = \rho B \partial \pm \partial = \pm \mathrm{sech}\beta \cdot e^{\pm\beta}\partial, \tag{5.3}$$

it follows that

$$(X, \chi_l)_{PB} = 1 - e^{2\beta_l}\Delta_l, \tag{5.4}$$

$$(\cosh\beta \cdot \hat{P}, \chi_l)_{PB} = e^{\beta}\partial + e^{-\beta}\partial e^{2\beta_l}\Delta_l = (e^{\beta} - e^{-\beta}e^{2\beta_l}\Delta_l)\partial. \tag{5.5}$$

Then, defining $\tilde{P}$ by

$$\tilde{P} \equiv \cosh\beta \cdot \hat{P}, \tag{5.6}$$

we get, from (5.1), a formula

$$(\tilde{P}, \chi_l)_{PB} = (1 - \Delta_l)e^{\beta}\partial. \tag{5.7}$$

In the following, by using Eqs.(5.4), (5.7) and (4.21) we calculate Dirac brackets for X and $\tilde{P}$.

1. $(X, X)_{DB}$

First let us calculate

$$\begin{aligned}(X, X)_{DB} &= -\sum_{l,l'} (X, X_l)_{PB}(G^{-1})_{ll'}(\chi_{l'}, X)_{PB} \\ &= \sum_{l,l'} (1 - e^{2\beta_l}\Delta_l)\partial^{-1}(\bar{\Lambda}_\gamma H^{-1} + \frac{1}{4}\Lambda_\gamma u u^T)_{ll'}(1 - e^{-2\beta_{l'}}\Delta_{l'}). \end{aligned} \tag{5.8}$$

The contributions from the constant term of ∂^{-1} can be shown to be $2c$, if one notices that $\Delta_l = \boldsymbol{I} = 1$ for any constant function. The other term $(1/2)\epsilon$ anti-commutes with Δ_l since ϵ is an odd function. The $\bar{\Lambda}_\gamma$ term becomes, by using Eqs.(4.18), (4.19) and (4.22), as follows:

$$\sum_{l'} \bar{\Lambda}_\gamma (H^{-1})_{ll'}(1 - e^{-2\beta_{l'}}\Delta_{l'}) = \bar{\Lambda}_\gamma (1 - e^{-2\pi\gamma}\boldsymbol{I})^{-1} \begin{pmatrix} -e^{-2\pi\gamma}\boldsymbol{I} - e^{-2\beta_0}\Delta_0 \\ 1 + e^{-2\beta_0}\Delta_0 \end{pmatrix}. \tag{5.9}$$

Also, from (4.24)

$$(1 - e^{-2\beta_l}\Delta_l)\bar{\Lambda}_\gamma = \bar{\Lambda}_\gamma - \bar{\Lambda}_{-\gamma}e^{-2\beta_l}\Delta_l, \tag{5.10}$$

and the contribution from $(1/2)\epsilon$ is

$$\begin{aligned} &\begin{pmatrix} \bar{\Lambda}_\gamma - \bar{\Lambda}_{-\gamma}e^{2\beta_0}\Delta_0, & \bar{\Lambda}_\gamma - \bar{\Lambda}_{-\gamma}e^{2\beta_\pi}\Delta_\pi \end{pmatrix} (1 - e^{-2\pi\gamma}\boldsymbol{I})^{-1} \begin{pmatrix} -e^{-2\pi\gamma}\boldsymbol{I} + e^{-2\beta_0}\Delta_0 \\ 1 - e^{-2\beta_0}\Delta_0 \end{pmatrix} \frac{\epsilon}{2} \\ &= \bar{\Lambda}_\gamma (1 - e^{-2\pi\gamma}\boldsymbol{I})^{-1}(1 - e^{-2\pi\gamma}\boldsymbol{I})\frac{\epsilon}{2} - \bar{\Lambda}_{-\gamma}(1 - e^{-2\pi\gamma}\boldsymbol{I}^{-1})^{-1}(1 - e^{-2\pi\gamma}\boldsymbol{I}^{-1})\frac{\epsilon}{2} \\ &= (\bar{\Lambda}_\gamma - \bar{\Lambda}_{-\gamma})\frac{\epsilon}{2} = -(\Lambda_\gamma - \Lambda_{-\gamma})\frac{\epsilon}{2}. \end{aligned} \tag{5.11}$$

Next, the Λ_γ terms are

$$\Lambda_\gamma e^{-2\beta_\pi}\Delta_\pi = \Lambda_\gamma e^{2\pi\gamma}\boldsymbol{I}^{-1}e^{-2\beta_0}\Delta_0 = \Lambda_\gamma e^{-2\beta_0}\Delta_0, \tag{5.12}$$

$$e^{2\beta_\pi}\Delta_\pi\Lambda_\gamma = e^{2\beta_0}\Delta_0 e^{-2\pi\gamma}\boldsymbol{I}\Lambda_\gamma = e^{2\beta_0}\Delta_0\Lambda_\gamma. \tag{5.13}$$

Hence

$$\begin{aligned}\frac{1}{2}(1-e^{2\beta_0}\Delta_0)\Lambda_\gamma(1+e^{-2\beta_0}\Delta_0)\frac{\epsilon}{2} &= \frac{1}{2}(\Lambda_\gamma - \Lambda_{-\gamma}e^{2\beta_0}\Delta_0)(1+e^{-2\beta_0}\Delta_0)\frac{\epsilon}{2}\\ &= \frac{1}{2}\{\Lambda_\gamma(1+e^{-2\beta_0}\Delta_0) - \Lambda_{-\gamma}(1+e^{2\beta_0}\Delta_0)\}\frac{\epsilon}{2}.\end{aligned} \tag{5.14}$$

Collecting all together we finally obtain

$$(X,X)_{DB} = -\frac{1}{2}\{\Lambda_\gamma(1-e^{-2\beta_0}\Delta_0) - \Lambda_{-\gamma}(1-e^{2\beta_0}\Delta_0)\}\frac{1}{2}\epsilon + 2c|s\rangle\langle s|, \tag{5.15}$$

where

$$|s\rangle \equiv \int_{-\infty}^{\infty} d\sigma|\sigma\rangle, \qquad \langle s|s\rangle = 1.$$

2. $(\tilde{P},X)_{DB}$

This is given by

$$\begin{aligned}(\tilde{P},X)_{DB} &= (\tilde{P},X)_{PB} - \sum_{ll'}(\tilde{P},\chi_l)_{PB}(G^{-1})_{ll'}(\chi_{l'},X)_{PB}\\ &= -2\cosh\beta + \sum_{ll'}(1-\Delta_l)e^{\beta}(\bar{\Lambda}_\gamma H^{-1} + \frac{1}{4}\Lambda_\gamma uu^T)_{ll'}(1-e^{-2\beta_{l'}}\Delta_{l'}).\end{aligned} \tag{5.16}$$

The use of (5.1) results in

$$\boldsymbol{I}\beta = (\beta - 2\pi\gamma)\boldsymbol{I}, \tag{5.17}$$

hence

$$\begin{aligned}e^{\beta}e^{-2\pi\gamma}\boldsymbol{I} &= \boldsymbol{I}e^{\beta},\\ e^{\beta}\Lambda_\gamma &= \Lambda_0 e^{\beta}\end{aligned} \tag{5.18}$$

and

$$\Delta_l(1-\boldsymbol{I})^{-1} = (1-\boldsymbol{I}^{-1})^{-1}\Delta_l = -(1-\boldsymbol{I})^{-1}\boldsymbol{I}\Delta_l. \tag{5.19}$$

These relations serve to calculate the $\bar{\Lambda}_\gamma$ term in (5.16)

$$\bar{\Lambda}_0(1-\boldsymbol{I})^{-1}\begin{pmatrix} 1+\Delta_0\boldsymbol{I}^{-1}, & 1+\Delta_0 \end{pmatrix}\begin{pmatrix} -\boldsymbol{I}e^{\beta} - \Delta_0 e^{-\beta} \\ e^{\beta} + \Delta_0 e^{-\beta}\end{pmatrix} = 2\bar{\Lambda}_0\cosh\beta. \tag{5.20}$$

and the Λ_γ term

$$\frac{1}{2}\Lambda_0(1-\Delta_0)(e^{\beta} - \Delta_0 e^{-\beta}) = \Lambda_0(1-\Delta_0)\cosh\beta. \tag{5.21}$$

Thus we get

$$(\tilde{P}, X)_{DB} = -\Lambda_0(1+\Delta_0)\cosh\beta. \tag{5.22}$$

3. $(\tilde{P}, \tilde{P})_{DB}$

This is given by

$$\begin{aligned}(\tilde{P}, \tilde{P})_{DB} &= -2\beta' - \sum_{ll'}(\tilde{P}, \chi_l)_{PB}(G^{-1})_{ll'}(\chi_{l'}, \tilde{P})_{PB} \\ &= -2\beta' - \sum_{ll'}(1-\Delta_l)e^{\beta}(\bar{\Lambda}_\gamma H^{-1} + \frac{1}{4}\Lambda_\gamma uu^T)_{ll'}\partial e^{-\beta}(1-\Delta_{l'}) \\ &= -2\beta' - \sum_{ll'}(1-\Delta_l)(\bar{\Lambda}_0 e^{\beta} H^{-1} + \frac{1}{4}\Lambda_0 e^{\beta} uu^T)_{ll'}\partial e^{-\beta}(1-\Delta_{l'}). \end{aligned} \tag{5.23}$$

The $\bar{\Lambda}_\gamma$ term becomes

$$\begin{aligned}&-\bar{\Lambda}_0(1-\Delta_0, 1-\Delta_\pi)e^{\beta}D_\gamma^{-1}\begin{pmatrix} 2 & -1-e^{-2\pi\gamma}\boldsymbol{I} \\ -1-e^{2\pi\gamma}\boldsymbol{I}^{-1} & 2 \end{pmatrix}\partial e^{-\beta}\begin{pmatrix} 1-\Delta_0 \\ 1-\Delta_\pi \end{pmatrix} \\ &= -\bar{\Lambda}_0 D_0^{-1}\begin{pmatrix} 1-\Delta_0, & 1-\Delta_\pi \end{pmatrix}\begin{pmatrix} 2 & -1-\boldsymbol{I} \\ -1-\boldsymbol{I}^{-1} & 2 \end{pmatrix} e^{\beta}\partial e^{-\beta}\begin{pmatrix} 1-\Delta_0 \\ 1-\Delta_\pi \end{pmatrix} \end{aligned} \tag{5.24}$$

and

$$e^{\beta}\partial e^{-\beta}\begin{pmatrix} 1-\Delta_0 \\ 1-\Delta_\pi \end{pmatrix} = (\partial - \beta')\begin{pmatrix} 1-\Delta_0 \\ 1-\Delta_\pi \end{pmatrix} = \begin{pmatrix} 1+\Delta_0 \\ 1+\Delta_\pi \end{pmatrix}\partial - \begin{pmatrix} 1-\Delta_0 \\ 1-\Delta_\pi \end{pmatrix}\beta'. \tag{5.25}$$

Hence, the $\bar{\Lambda}_\gamma$ term results in

$$2\bar{\Lambda}_0\beta'. \tag{5.26}$$

The other Λ_γ term becomes

$$\begin{aligned}-\frac{1}{2}\Lambda_0(1-\Delta_0)e^{\beta}\partial e^{-\beta}(1-\Delta_0) &= -\frac{1}{2}\Lambda_0(1-\Delta_0)(\partial-\beta')(1-\Delta_0) \\ &= \Lambda_0(1-\Delta_0)\beta'. \end{aligned} \tag{5.27}$$

Finally we have

$$(\tilde{P}, \tilde{P})_{DB} = -\Lambda_0(1+\Delta_0)\beta'. \tag{5.28}$$

VI. Quasi-periodic space

In this section we give a σ-dependence of the Dirac brackets.

Let the subspace $\boldsymbol{I} = e^{2\pi\gamma}$ be V_γ, which is projected by Λ_γ. This is a function space satisfying the quasi-periodicity

$$f(\sigma + 2\pi) = e^{2\pi\gamma} f(\sigma). \tag{6.1}$$

The element f of this space can be expanded into the Fourier-like series

$$f(\sigma) = \sum_n e^{(\gamma - in)\sigma} f_n, \quad (f \in V_\gamma) \tag{6.2}$$

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma e^{-(\gamma - in)\sigma} f(\sigma). \quad (f \in V_\gamma) \tag{6.3}$$

Therefore, we have the expression

$$\begin{aligned} \langle \sigma | \Lambda_\gamma | \sigma' \rangle &= \frac{1}{2\pi} \sum_n e^{(\gamma - in)(\sigma - \sigma')} = e^{\gamma(\sigma - \sigma')} \bar{\delta}(\sigma - \sigma') \\ &= \sum_k e^{2k\pi\gamma} \delta(\sigma - \sigma' - 2k\pi), \end{aligned} \tag{6.4}$$

where $\bar{\delta}$ is the periodic delta function. As for any function f which does not belong to V_γ , we get, using the Fourier integral

$$f(\sigma) = \int_{-\infty}^{\infty} dk e^{-ik\sigma} \tilde{f}(k), \tag{6.5}$$

the following expansion:

$$\begin{aligned} \langle \sigma | \Lambda_\gamma | f \rangle &= \int_{-\infty}^{\infty} d\sigma' \langle \sigma | \Lambda_\gamma | \sigma' \rangle \int_{-\infty}^{\infty} dk e^{-ik\sigma'} \tilde{f}(k) \\ &= \sum_n e^{(\gamma - in)\sigma} \int_{-\infty}^{\infty} dk \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma' e^{-i(k - n - i\gamma)\sigma'} \tilde{f}(k) \\ &= \sum_n e^{(\gamma - in)\sigma} \int_{-\infty}^{\infty} dk \delta(k - n - i\gamma) \tilde{f}(k) \end{aligned} \tag{6.6}$$

that is,

$$\langle \sigma | \Lambda_\gamma | f \rangle = \sum_n e^{(\gamma - in)\sigma} \tilde{f}(n + i\gamma). \tag{6.7}$$

Here we have used a fact that eigenvalues of $i\gamma$ are real for space components of γ. (As for time components of γ, it may be necessary to consider Euclidization.)

As for $\Lambda_\gamma \epsilon$ in (5.14), we use the Fourier integral of $\epsilon(\sigma)$

$$\epsilon(\sigma - \sigma') = \frac{i}{\pi} \int_{-\infty}^{\infty} dk e^{-ik(\sigma - \sigma')} \frac{k}{k^2 + \epsilon^2} \tag{6.8}$$

to give

$$\begin{aligned} \langle \sigma | \Lambda_\gamma \epsilon | \sigma' \rangle &= \frac{1}{2\pi^2 i} \sum_n e^{(\gamma - in)(\sigma - \sigma')} \int_{-\infty}^{\infty} dk \frac{k}{k^2 + \epsilon^2} \int_{-\infty}^{\infty} d\sigma'' e^{i(k + n + i\gamma)\sigma''} \\ &= \frac{i}{\pi} \sum_n e^{(\gamma - in)(\sigma - \sigma')} \frac{n + i\gamma}{(n + i\gamma)^2 + \epsilon^2}. \end{aligned} \tag{6.9}$$

If the eigenvalue of $i\gamma$ is not integer, this defines the function

$$\bar{\epsilon}(\sigma - \sigma' : \gamma) \equiv \langle \sigma | \Lambda_\gamma \epsilon | \sigma' \rangle = \frac{1}{\pi} \sum_n \frac{1}{\gamma - in} e^{(\gamma - in)(\sigma - \sigma')}. \tag{6.10}$$

In terms of this function Eq.(5.15) can be written as

$$\langle\sigma|(X,X)_{DB}|\sigma'\rangle = (X(\sigma),X(\sigma'))_{DB} = -\frac{1}{4}E(\sigma,\sigma':\gamma)+2c, \tag{6.11}$$

where

$$E(\sigma,\sigma':\gamma) \equiv \bar{\epsilon}(\sigma-\sigma':\gamma)+\bar{\epsilon}(-\sigma+\sigma':\gamma)+e^{-2\beta_0}\bar{\epsilon}(\sigma+\sigma':\gamma)+e^{2\beta_0}\bar{\epsilon}(-\sigma-\sigma':\gamma). \tag{6.12}$$

In (5.22) and (5.28) we have

$$\langle\sigma|\Lambda_0(1+\Delta_0)|\sigma'\rangle = \bar{\delta}(\sigma-\sigma')+\bar{\delta}(\sigma+\sigma') = \delta_c(\sigma,\sigma'), \tag{6.13}$$

hence

$$(\tilde{P}(\sigma),X(\sigma'))_{DB} = -\delta_c(\sigma,\sigma')\cosh\beta(\sigma'), \tag{6.14}$$

$$(\tilde{P}(\sigma),\tilde{P}(\sigma'))_{DB} = -\delta_c(\sigma,\sigma')\beta'(\sigma') = \gamma\delta_c(\sigma,\sigma'). \tag{6.15}$$

The quantized theory is given by replacing the Dirac bracket with the commutator

$$(A,B)_{DB} \;\rightarrow\; -i[A,B]. \tag{6.16}$$

We summarize relevant commutation relations:

$$[X(\sigma),\tilde{P}(\sigma')] = i\cosh\beta(\sigma)\delta_c(\sigma,\sigma'), \tag{6.17}$$

$$[\tilde{P}(\sigma),\tilde{P}(\sigma')] = i\gamma\delta_c(\sigma,\sigma'), \tag{6.18}$$

$$[X(\sigma),P(\sigma')] = i\{\delta_c(\sigma,\sigma')-\frac{1}{2}\gamma\mathrm{sech}^2\beta(\sigma)[X(\sigma),X(\sigma')]\}, \tag{6.19}$$

$$[P(\sigma),P(\sigma')] = \gamma\mathrm{sech}^2\beta(\sigma)[X(\sigma),X(\sigma')]\gamma\mathrm{sech}^2\beta(\sigma'), \tag{6.20}$$

$$[X(\sigma),X(\sigma')] = -\frac{i}{4}E(\sigma,\sigma':\gamma)+i2c. \tag{6.21}$$

where

$$2c = \frac{\cosh\beta_0\cosh\beta_\pi}{\sinh\pi\gamma}. \tag{6.22}$$

In the Appendix A we derive the explicit form of E and give the proof of Eq.(6.22). From Eq.(6.21) one finds noncommutativity of X at both ends

$$[X(0),X(0)] = -i\cosh\beta_0\sinh\beta_0, \tag{6.23}$$

$$[X(\pi),X(\pi)] = i\cosh\beta_\pi\sinh\beta_\pi. \tag{6.24}$$

According to these equations we see that noncommutativity reveals also to $[P,P]$ and $[X,P]$.

VII. Mode expansions

In this section we consider mode expansions of relevant fields.

The constraint (4.5) leads to

$$P_+ = e^{-2\pi\gamma}\boldsymbol{I}P_+ \qquad \in V_\gamma, \tag{7.1}$$

$$P_- = e^{2\beta_0}\Delta_0 P_+ \qquad \in V_{-\gamma}. \tag{7.2}$$

Hence Fourier-like expansions of them are

$$P_+(\sigma) = \frac{1}{2\sqrt{\pi}}\sum_n e^{(\gamma-in)\sigma-\beta_0}\alpha_n, \tag{7.3}$$

$$P_-(\sigma) = \frac{1}{2\sqrt{\pi}}\sum_n e^{-\{(\gamma-in)\sigma-\beta_0\}}\alpha_n. \tag{7.4}$$

From the definition (4.3) of $P_\pm$, therefore, we have mode expansions

$$\begin{aligned}\dot{X}(\sigma) = P_+(\sigma) + P_-(\sigma) &= \frac{1}{\sqrt{\pi}}\sum_n \cosh\{(\gamma-in)\sigma-\beta_0\}\alpha_n \\ &= \cosh\beta(\sigma)\cdot\tilde{P}(\sigma) - \sinh\beta(\sigma)\cdot\tilde{Q}(\sigma),\end{aligned} \tag{7.5}$$

$$\begin{aligned}X'(\sigma) = P_+(\sigma) - P_-(\sigma) &= \frac{1}{\sqrt{\pi}}\sum_n \sinh\{(\gamma-in)\sigma-\beta_0\}\alpha_n \\ &= -\sinh\beta(\sigma)\cdot\tilde{P}(\sigma) + \cosh\beta(\sigma)\cdot\tilde{Q}(\sigma),\end{aligned} \tag{7.6}$$

where

$$\tilde{P}(\sigma) = \cosh\beta(\sigma)\cdot\dot{X}(\sigma) + \sinh\beta(\sigma)\cdot X'(\sigma) = \frac{1}{\sqrt{\pi}}\sum_n \cos n\sigma\cdot\alpha_n, \tag{7.7}$$

$$\tilde{Q}(\sigma) = \sinh\beta(\sigma)\cdot\dot{X}(\sigma) + \cosh\beta(\sigma)\cdot X'(\sigma) = -i\frac{1}{\sqrt{\pi}}\sum_n \sin n\sigma\cdot\alpha_n. \tag{7.8}$$

From the equation of motion (2.5) for X, one can see the τ-dependence of α_n to be

$$\alpha_n(\tau) = e^{(\gamma-in)\tau}\alpha_n(0) = \alpha_n(0)e^{(-\gamma-in)\tau}. \tag{7.9}$$

The mode expansion of X, is, therefore, given by

$$X(\tau,\sigma) = \frac{1}{\sqrt{\pi}}\sum_n \frac{1}{\gamma-in}e^{(\gamma-in)\tau}\cosh\{(\gamma-in)\sigma-\beta_0\}\alpha_n(0) + \frac{1}{\sqrt{\pi}}b, \tag{7.10}$$

and

$$P(\tau,\sigma) = \mathrm{sech}\beta(\sigma)\tilde{P}(\tau,\sigma) + \frac{1}{2}\gamma\mathrm{sech}^2\beta(\sigma)X(\tau,\sigma). \tag{7.11}$$

The over-all time-development factor $e^{\gamma\tau}$ in (7.10) means that $i\gamma$ (or its positive eigenvalue) is the cyclotron frequency of the string. The commutation relation for the mode operators α_n is found to be

$$[\alpha_m, \alpha_n] = (m + i\gamma)\delta_{m+n,0}, \tag{7.12}$$

owing to (6.17) and (6.18).

VIII. Virasoro algebra

The Virasoro operator is defined by (see Appendix B for note)

$$L_n = \frac{1}{4}\int_{-\pi}^{\pi} d\sigma e^{\pm in\sigma} :(\dot{X} \pm X')^2 := \frac{1}{2}\sum_k :\alpha_k \cdot \alpha_{n-k}:, \tag{8.1}$$

which satisfies the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + m(am^2 - b)\delta_{m+n,0}, \tag{8.2}$$

where

$$a = \frac{1}{12}d, \qquad b = \frac{1}{12}d - \frac{1}{2}\mathrm{tr}(\gamma^2). \qquad (d = \text{space-time dimension}) \tag{8.3}$$

The central term differs from the conventional one owing to the unconventional commutation relation (7.12). This conforms the well-known result in Ref.[9]. The extra term $-\mathrm{tr}(\gamma^2)/2$ can be eliminated by a shift $L_0 \to L_0 + \mathrm{tr}(\gamma^2)/4$.

IX. Dependence on the cyclotron frequency

1. The periodicity on the parameters

In view of (2. 10) one sees that β_0 (β_π) may differ by an integral multiple of $i\pi$ to give the same $\rho(0)$ ($\rho(\pi)$). This means that the system is π-periodic with respect to $i\beta_0$ and $i\pi\gamma$. In this section we would like to investigate how these periodicities are related to the mode operators.

First of all, Eq. (7. 10) tells us that the displacement of β_0

$$\beta_0 \to \beta_0 + in\pi \tag{9.1}$$

cause the change in the signature of α_n:

$$\alpha_n \to (-1)^n \alpha_n \tag{9.2}$$

To investigate the periodicity in γ it is convenient to represent B in two-dimensional blocks. For definiteness we consider the spatial components (related to the magnetic field). Then the i-th block has a form

$$B^{(i)} = \lambda^{(i)}\epsilon = \lambda^{(i)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9.3}$$

We represent also β_0, β_π and γ as

$$\beta_0^{(i)} = \pi\omega_0^{(i)}\epsilon, \qquad \beta_\pi^{(i)} = \pi\omega_\pi^{(i)}\epsilon, \qquad \gamma^{(i)} = \omega^{(i)}\epsilon \tag{9.4}$$

where

$$\omega^{(i)} = \omega_0^{(i)} - \omega_\pi^{(i)} \tag{9.5}$$

is the cyclotron frequency mentioned in sect.VII. Henceforth we confine ourselves to one particular block and suppress the index i.

We adopt a complex basis in the two-dimensional subspace

$$e^{\pm} = \frac{1}{\sqrt{2}}(1, \pm i) = \frac{1}{\sqrt{2}}(e_1 \pm ie_2), \tag{9.6}$$

which diagonalizes ϵ

$$\epsilon \cdot e^{\pm} = \pm ie^{\pm}, \qquad e^{\pm} \cdot \epsilon = \mp ie^{\pm}. \tag{9.7}$$

Then from (7. 10) we have

$$\begin{aligned} X^{\pm}(\tau, \sigma; \omega) &= e^{\pm} \cdot X(\tau, \sigma; \omega) \\ &= \frac{1}{\sqrt{\pi}} \sum_n \frac{i}{n \pm \omega} e^{-i(n\pm\omega)\tau} \cos\{(n \pm \omega)\sigma - \pi\omega_0\} \alpha_n^{\pm}(\omega) + \frac{1}{\sqrt{\pi}} b^{\pm}(\omega). \end{aligned} \tag{9.8}$$

We have explicitly written the ω-dependence of the operators. This expansion essentially coincides with that in Ref.[9].

The mode operators in (9. 8) satisfy

$$\alpha_n^{\pm}(\omega)^{\dagger} = \alpha_{-n}^{\mp}(\omega), \qquad b^{\pm}(\omega)^{\dagger} = b^{\mp}(\omega) \tag{9.9}$$

and their non-vanishing commutators are

$$[\alpha_m^{\pm}(\omega), \alpha_n^{\mp}(\omega)] = (m \pm \omega)\delta_{m+n,0}, \tag{9.10}$$

$$[b^{\pm}(\omega), b^{\mp}(\omega)] = -\pi \frac{\cos \pi\omega_0 \cos \pi\omega_\pi}{\sin \pi\omega}. \tag{9.11}$$

The periodicity on γ requires $X^{\pm}(\tau, \sigma; \omega + l) = X^{\pm}(\tau, \sigma; \omega)$ for any integer l, which implies

$$\alpha_n^{\pm}(\omega + l) = \alpha_{n\pm l}^{\pm}(\omega) \tag{9.12}$$

and

$$b^{\pm}(\omega + l) = b^{\pm}(\omega). \tag{9.13}$$

These relations are easily seen to be consistent with (9. 10) and (9. 11), if one notes the equality $\omega_\pi = \omega_0 - \omega$. The relation (9. 12) determines the ω-dependence of the $\alpha_n^{\pm}(\omega)$'s as

$$\alpha_n^{\pm}(\omega) = \alpha_0^{\pm}(\omega \pm n). \tag{9.14}$$

2. The limit to the neutral string

If ω tends to an integral value m, Eqs. (2. 11) and (2. 10) give $\rho(\pi) \to \rho(0)$ and we have the neutral string. Let us see how this occurs for the mode operators.

In Eq. (9. 8) the term with $n = \pm m$ becomes singular as $\omega \to m$. This singularity should be cancelled by a similar singularity of $b^{\pm}(\omega)$.

It is sufficient to consider the case $m = 0$. To retain the symmetry between the two end points we define

$$\bar{\beta} \equiv \frac{1}{2}(\beta_0 + \beta_\pi), \qquad \beta_0 = \bar{\beta} + \frac{1}{2}\pi\gamma, \qquad \beta_\pi = \bar{\beta} - \frac{1}{2}\pi\gamma. \tag{9.15}$$

The commutation relations for α_0 and b are

$$[\alpha_0, \alpha_0] = i\gamma, \tag{9.16}$$
$$[\alpha_0, b] = 0 \tag{9.17}$$

and

$$[b, b] = i\gamma^{-1} \cosh^2 \bar{\beta} + O(\gamma). \tag{9.18}$$

If we expand α_0 as

$$\alpha_0{}^i \approx u^i + \frac{1}{2}(\gamma \cdot v)^i \qquad (\gamma \approx 0), \tag{9.19}$$

from (9. 16) we get,

$$[u^i, u^j] = 0, \qquad [u^i, v^j] = -i\delta^{ij}. \tag{9.20}$$

The $n = 0$ term in (7.10) is reduced to, apart from the factor $\frac{1}{\sqrt{\pi}}$, to

$$\begin{aligned} \gamma^{-1} e^{\gamma\tau} \cosh(\gamma\sigma - \beta_0)\alpha_0 &= \cosh\beta_0 \cdot (\gamma^{-1}u + \tau u + \frac{v}{2}) - \sigma \sinh\beta_0 \cdot u + O(\gamma) \\ &= \cosh\bar{\beta} \cdot (\gamma^{-1}u + \tau u + \frac{v}{2}) + (\frac{\pi}{2} - \sigma) \sinh\bar{\beta} \cdot u + O(\gamma). \end{aligned} \tag{9.21}$$

On the other hand for the neutral string we have

$$\begin{aligned} X(\tau, \sigma : \gamma = 0) = x + \frac{1}{\sqrt{\pi}} \{\tau \cosh\bar{\beta} + (\frac{\pi}{2} - \sigma) \sinh\bar{\beta}\} \cosh\bar{\beta} \cdot p \\ + \frac{1}{\sqrt{\pi}} \sum_{n \neq 0} \frac{i}{n} e^{-in\tau} \cos(n\sigma - i\bar{\beta})\alpha_n(\gamma = 0) \end{aligned} \tag{9.22}$$

where (x, p) is the canonical pair. It can be shown that the $X(\tau, \sigma)$ in (7. 10) tends to (9. 22) and the commutation relations (9. 17) and (9. 18) hold in the lowest order in γ, if

$$u = \frac{1}{\sqrt{\pi}} \cosh\bar{\beta} \cdot p, \qquad v = \sqrt{\pi}\mathrm{sech}\bar{\beta} \cdot x \tag{9.23}$$

and

$$b = -\cosh\bar{\beta} \cdot (\gamma^{-1} \cdot u - \frac{v}{2}) + O(\gamma). \tag{9.24}$$

X. The g-factors of charged strings

In this section we calculate the g-factor of the charged string. The magnetic moment $\boldsymbol{\mu}$ of the system is defined, in the non-relativistic limit, as,

$$\boldsymbol{\mu} = -\frac{\partial E}{\partial \boldsymbol{B}}\Big|_{B=0} = \frac{1}{2E} \frac{\partial p^2}{\partial \boldsymbol{B}}\Big|_{B=0} \approx \frac{1}{2M} \frac{\partial p^2}{\partial \boldsymbol{B}}\Big|_{B=0}, \tag{10.1}$$

where E is the energy, p the four-momentum and M the mass of the particle.

In the case of our charged string the four-momentum squared is fixed by the Virasoro on-shell condition

$$\{L_0 - \alpha(0)\}|\Psi\rangle = 0 \tag{10.2}$$

where

$$\alpha(0) = 1 - \frac{1}{4}\mathrm{tr}(\gamma^2) \tag{10.3}$$

and

$$\begin{aligned} L_0 &= \frac{1}{2}\int_0^\pi d\sigma(\dot{X}^2 + X'^2) \\ &= \frac{1}{2}\int_0^\pi d\sigma\{(P - \rho B X' - \frac{1}{2}\rho' B X)^2 + X'^2\}. \end{aligned} \tag{10.4}$$

For a weak magnetic field $\boldsymbol{B}$ Eq.(10. 4) becomes

$$\{L_0|_{B=0} - 1 - \boldsymbol{B}\cdot\boldsymbol{G} + O(\boldsymbol{B}^2)\}|\Psi\rangle = 0, \tag{10.5}$$

where

$$L_0|_{B=0} = (\sum_{n>0}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\alpha_0^2)|_{\boldsymbol{B}=0} = R + \frac{1}{2\pi}p^2 \tag{10.6}$$

and

$$\boldsymbol{G} = -\frac{\partial L_0}{\partial \boldsymbol{B}}\Big|_{B=0} = \boldsymbol{G}_1 + \boldsymbol{G}_2 \tag{10.7}$$

with

$$\boldsymbol{G}_1 = -\frac{1}{2}\int_0^\pi d\sigma(\rho'\boldsymbol{X}\times\boldsymbol{P})\Big|_{B=0} \tag{10.8}$$

and

$$\boldsymbol{G}_2 = -\int_0^\pi d\sigma(\rho\boldsymbol{X}'\times\boldsymbol{P})\Big|_{B=0}. \tag{10.9}$$

We consider the case that $|\Psi\rangle$ is a particle state with definite values of R and p^2 . The quantity R is related to the mass operator M of the particle through

$$R - 1 = \frac{1}{2\pi}M^2. \tag{10.10}$$

We have therefore

$$[\frac{1}{2\pi}(p^2 + M^2) - \boldsymbol{B}\cdot\boldsymbol{G}]|\Psi\rangle = 0, \tag{10.11}$$

which in view of Eq. (10. 1), gives

$$\boldsymbol{\mu} = \frac{\pi}{M}\boldsymbol{G}. \tag{10.12}$$

Before calculating G we note that in the limit $\boldsymbol{B}\to 0$ the function $\rho(\sigma)$ becomes

$$\rho(\sigma) \approx \pi^{-1}\{\rho(\pi) - \rho(0)\}\sigma + \rho(0) = q_0 - \pi^{-1}q\sigma \tag{10.13}$$

with the total charge

$$q = q_0 + q_\pi. \tag{10.14}$$

The quantity $\boldsymbol{G}_1$ is readily seen to be proportional to the total angular momentum

$$\boldsymbol{G}_1 = \frac{1}{2\pi} q \boldsymbol{J} = \frac{1}{2\pi} q(\boldsymbol{L} + \boldsymbol{S}) \tag{10.15}$$

with the orbital and spin angular momenta

$$\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}, \tag{10.16}$$

$$\boldsymbol{S} = \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \boldsymbol{\alpha_n} \times \boldsymbol{\alpha_{-n}}. \tag{10.17}$$

The integration in (10. 9) can be performed to give

$$\boldsymbol{G}_2 = \frac{i}{2\pi} \sum_{n \neq n'} \boldsymbol{\alpha_n} \times (\boldsymbol{\alpha_{n'}} - \boldsymbol{\alpha_{-n'}}) \frac{q_0 + (-)^{n-n'} q_\pi}{n - n'}. \tag{10.18}$$

If the state $|\Psi\rangle$ is an eigenstate of R, only the terms of the form $\boldsymbol{\alpha_n} \times \boldsymbol{\alpha_{-n}}$ contribute to the expectation value of $\boldsymbol{G}_2$:

$$\langle \Psi | \boldsymbol{G}_2 | \Psi \rangle = \frac{i}{2\pi} \sum_{n \neq 0} \frac{q_0 + q_\pi}{2n} \langle \Psi | \boldsymbol{\alpha_n} \times \boldsymbol{\alpha_{-n}} | \Psi \rangle = \frac{q}{2\pi} \langle \Psi | \boldsymbol{S} | \Psi \rangle. \tag{10.19}$$

As a result we get

$$\langle \Psi | \boldsymbol{\mu} | \Psi \rangle = \frac{q}{2M} \langle \Psi | (\boldsymbol{L} + 2\boldsymbol{S}) | \Psi \rangle \tag{10.20}$$

so that the g-factors are the same as for the charged Dirac particle.

$$g_L = 1 \qquad \text{and} \qquad g_S = 2. \tag{10.21}$$

We note that any change of the function $\rho(\sigma) \to \rho(\sigma) + \delta\rho(\sigma)$ does not affect this result, since the expectation value of

$$\delta \boldsymbol{G} = -\frac{1}{2} \int_0^\pi d\sigma \delta\rho(\boldsymbol{X}' \times \boldsymbol{P} - \boldsymbol{X} \times \boldsymbol{P}') \tag{10.22}$$

is seen to vanish.

XI. Conclusions

We have performed the Dirac quantization for constrained system of the charged string in the constant B field. Noncommutativity appears at both ends of X and also of P. We have considered the dependence of the cyclotron frequency of the charged string. We have found that a displacement of the cyclotron frequency $\omega \to \omega + l$ (l: integer) causes the mode translation $n \to n \pm l$ in $\alpha_n^\pm$. When ω takes an integral value, we have the neutral string. We have seen how this occurs for the mode operastors. We have finally calculated the g-factors of charged strings in the framework of our formulation, which leads us to the noncommutative strings. We have found $g_s = 2$ for the string with any spin s, and $g_L = 1$ for the same string with any orbital angular momentum L.

Appendix A

In this Appendix we derive more explicit forms of functions $\bar{\epsilon}(\sigma:\gamma)$, $E(\sigma,\sigma':\gamma)$ and also of $[b,b]$.

The function $\bar{\epsilon}(\sigma:\gamma)$ is defined by

$$\bar{\epsilon}(\sigma:\gamma) \equiv \frac{1}{\pi}\sum_n \frac{1}{\gamma - in} e^{(\gamma-in)\sigma}. \tag{A.1}$$

Hence we have

$$\bar{\epsilon}'(\sigma:\gamma) = \frac{1}{\pi}\sum_n e^{(\gamma-in)\sigma} = 2e^{\gamma\sigma}\delta(\sigma) \tag{A.2}$$

and

$$\begin{aligned} \bar{\epsilon}(0:\gamma) &= \frac{1}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{\gamma-in} = \frac{i}{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n+i\gamma} \\ &= i\cot(i\pi\gamma) = \coth\pi\gamma. \end{aligned} \tag{A.3}$$

Define

$$\begin{aligned} \hat{\epsilon}(\sigma:\gamma) \equiv 2\int_0^\sigma d\sigma' e^{\gamma\sigma'}\bar{\delta}(\sigma') &= 2\sum_{k=-\infty}^{\infty}\int_0^\sigma d\sigma' e^{2\pi k\gamma}\delta(\sigma'-2\pi k\gamma) \\ &= \sum_{k=-\infty}^{\infty} e^{2\pi k\gamma}\{\epsilon(\sigma-2k\pi)+\epsilon(2k\pi)\}, \end{aligned} \tag{A.4}$$

where

$$\epsilon(\sigma) \equiv \mathrm{sgn}(\sigma). \tag{A.5}$$

Then we get

$$\bar{\epsilon}(\sigma:\gamma) = \hat{\epsilon}(\sigma:\gamma) + \coth\pi\gamma. \tag{A.6}$$

For $2n\pi < \sigma < 2(n+1)\pi$, the first term in (A.4) can be written as

$$\begin{aligned} \sum_{k=-N}^{N} e^{2\pi k\gamma}\epsilon(\sigma-2k\pi) &= \sum_{k=-N}^{n} e^{2\pi k\gamma} - \sum_{k=n+1}^{N} e^{2\pi k\gamma} \\ &= \frac{1}{e^{2\pi\gamma}-1}\{e^{2(n+1)\pi\gamma} - e^{-2N\pi\gamma} - e^{2(N+1)\pi\gamma}\}. \end{aligned}$$

On the other hand, the second term is

$$\sum_{k=-N}^{N}\epsilon(2k\pi)e^{2\pi k\gamma} = \sum_{k=1}^{N}(e^{2\pi k\gamma}-e^{-2\pi k\gamma}) = \frac{e^{2(N+1)\pi\gamma}-e^{2\pi\gamma}+e^{-2N\pi\gamma}-1}{e^{2\pi\gamma}-1}.$$

Hence, we have

$$\begin{aligned} \hat{\epsilon}(\sigma:\gamma) &= \frac{1}{e^{2\pi\gamma}-1}\{2e^{2(n+1)\pi\gamma} - e^{2\pi\gamma} - 1\} \\ &= \frac{e^{(2n+1)\pi\gamma}}{\sinh\pi\gamma} - \coth\pi\gamma \qquad (2n\pi < \sigma < 2(n+1)\pi) \end{aligned} \tag{A.7}$$

to give

$$\bar{\epsilon}(\sigma:\gamma)=\frac{e^{(2n+1)\pi\gamma}}{\sinh\pi\gamma}. \qquad (2n\pi<\sigma<2(n+1)\pi) \tag{A.8}$$

At $\sigma = 2n$ this is defined by a mean value of both sides

$$\bar{\epsilon}(2n\pi:\gamma)=\frac{e^{(2n+1)\pi\gamma}+e^{(2n-1)\pi\gamma}}{2\sinh\pi\gamma}=e^{2n\pi\gamma}\coth\pi\gamma. \tag{A.9}$$

From (A.8) and (A.9) the explicit formula of $\epsilon(\sigma:\gamma)$ is

$$\bar{\epsilon}(\sigma:\gamma)=\frac{e^{\pi\gamma}}{\sinh\pi\gamma}\sum_k e^{2\pi k\gamma}\{\theta(\sigma-2k\pi)-\theta(\sigma-2(k+1)\pi)\}, \tag{A.10}$$

where θ is the ordinary step function.

The function $E(\sigma,\sigma':\gamma)$ is defined by (6.12), i.e.

$$\begin{aligned}E(\sigma,\sigma':\gamma)&\equiv\bar{\epsilon}(\sigma-\sigma':\gamma)+\bar{\epsilon}(-\sigma+\sigma':\gamma)\\&+e^{-2\beta_0}\bar{\epsilon}(\sigma+\sigma':\gamma)+e^{2\beta_0}\bar{\epsilon}(-\sigma-\sigma':\gamma).\end{aligned} \tag{A.11}$$

For $0<\sigma,\sigma'<\pi$, this becomes

$$E(\sigma,\sigma':\gamma)=\frac{2}{\sinh\pi\gamma}\{\cosh\pi\gamma+\cosh(\pi\gamma-2\beta_0)\}=\frac{4\cosh\beta_0\cosh\beta_\pi}{\sinh\pi\gamma}. \tag{A.12}$$

In order that X is commutable in this region, the constant term in (6.21) should be

$$i2c=\frac{1}{\pi}[b,b]=i\frac{\cosh\beta_0\cosh\beta_\pi}{\sinh\pi\gamma}. \tag{A.13}$$

This coincides with Eq.(6.22).

Appendix B

We summarize a consequence of the coefficient 1/4 in our Lagrangian.(2.4). For brevity we set $B=0$. The Lagrangian is given by

$$L=\frac{1}{4}\int_0^\pi d\sigma(\dot{X}^2-X'^2). \tag{B.1}$$

The canonical momentum is defined by

$$P_c\equiv\frac{\partial L}{\partial\dot{X}}=\frac{1}{2}\dot{X}. \tag{B.2}$$

The Poisson bracket and the Dirac bracket are then given by

$$(X(\sigma),P_c(\sigma'))_{PB}=\delta(\sigma-\sigma') \tag{B.3}$$

and

$$(X(\sigma), P_c(\sigma'))_{DB} = \frac{1}{2}\delta_c(\sigma, \sigma') \tag{B.4}$$

respectively. (Eq.(B.4) is obtained by taking $B \to 0$ in Eq.(6.19).)

We define the Virasoro operator as

$$\tilde{L}_n = \frac{1}{8}\int_{-\pi}^{\pi} d\sigma e^{\pm in\sigma}(\dot{X} \pm X')^2 = \frac{1}{2}\int_{-\pi}^{\pi} d\sigma e^{\pm in\sigma}(P_c \pm \frac{1}{2}X')^2, \tag{B.5}$$

which leads us to

$$(X(\sigma), \tilde{L}_n)_{PB} = \dot{X}(\sigma)\cos n\sigma + iX'(\sigma)\sin n\sigma \tag{B.6}$$

for the Poisson bracket, and to

$$(X(\sigma), \tilde{L}_n)_{DB} = \frac{1}{2}(\dot{X}(\sigma)\cos n\sigma + iX'(\sigma)\sin n\sigma) \tag{B.7}$$

for the Dirac bracket. In view of 1/2 in (B.7) one can see that the conformal generator in the Dirac quantization should be

$$L_n = 2\tilde{L}_n = \frac{1}{4}\int_{-\pi}^{\pi} d\sigma e^{\pm in\sigma} : (\dot{X} \pm X')^2 : . \tag{B.8}$$

To sum up, it is convenient to use the modified momentum

$$P \equiv 2P_c = \dot{X} \tag{B.9}$$

to give

$$(X(\sigma), P(\sigma'))_{DB} = \delta_c(\sigma, \sigma'), \tag{B.10}$$

$$L_n = \frac{1}{4}\int_{-\pi}^{\pi} d\sigma e^{\pm in\sigma}(\dot{X} \pm X')^2 = \frac{1}{4}\int_{-\pi}^{\pi} d\sigma e^{\pm in\sigma}(P \pm X')^2. \tag{B.11}$$

The results coincide with those in the conventional quantization.

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